On the global structure of Kerr-de Sitter spacetimes

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The global structure of the family of Kerr de Sitter spacetimes is re-examined. Taking advantage of the natural length scale set by the cosmological constant $\Lambda > 0$, conditions on the parameters (Λ, M, a^2) have been found, so that a Kerr-de Sitter specetime either describes a black hole with well separated horizons, or describes degenerate configurations where two or more horizons coincide. As long as the rotation parameter a^2 is subject to the constraint $a^2\Lambda \ll 1$, while the mass parameter M is subject to: $a^2[1+O(a^2\Lambda)^2)] < M^2 < \frac{1}{9\Lambda}[1+2a^2\Lambda+O(a^2\Lambda)^2)]$, then a Kerr-de Sitter spacetime with such parameters, describes a black hole possessing an inner horizon separated from an outer horizon and the hole is embedded within a pair of cosmological horizons. Still for $a^2\Lambda \ll 1$, but assuming that either $M^2 > \frac{1}{9\Lambda}[1+2a^2\Lambda+O(a^2\Lambda)^2)]$ or $M^2 < a^2[1+O(a^2\Lambda)^2)]$, the Kerr-de Sitter spacetime describes a ring-like singularity enclosed by two cosmological horizons. A Kerr-de Sitter spacetime may also describe configurations where the inner, the outer and one of the cosmological horizons coincide. However, we found that this coalescence occurs provided $M^2\Lambda \sim 1$ and due to the observed smallness of Λ, these configurations are probably irrelevant in astrophysical settings. Extreme black holes, i.e. black holes where the inner horizon coincides with the outer black hole horizon are also admitted. We have found that in the limit $M^2\Lambda \ll 1$ and $a^2\Lambda \ll 1$, extreme black holes occur, provided $a^2 = M^2(1+O(\Lambda M^2))$. Finally a coalescence between the outer and the cosmological horizon, although in principle possible, is likely to be unimportant at the astrophysical level, since this requires $M^2\Lambda \sim 1$. Our analysis shows that as far as the structure of the horizons are concerned, the family of Kerr-de Sitter spacetimes exhibits similar structure as the Reissner-Nordstrom-de Sitter family of spacetimes does.

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I. INTRODUCTION

Since the inception of General Theory of Relativity, the rise, fall and the eventual reemergence of the cosmological constant Λ has an interesting story[1]. In 1917, Einstein introduced the Λ term into his famous equations hoping that the repulsive effects associated with $\Lambda > 0$ would lead to a static universe. However since observational data favored a dynamical world model, he abandoned Λ a few years later. With the advent of spontaneous symmetry breaking in gauge theories, the Λ term re-appeared and now days is at the epicenter of one of the deep mysteries surrounding modern science. A multiple of Λ is interpreted as the vacuum energy density and the real issue is why? After so many symmetry breakings that took place in the early universe, does Λ relax to the tiny value suggested by current observations?

The recently discovered type-Ia cosmological supernovae provide direct observational evidences for a positive cosmological constant and these developments brought Λ back into the forefront of scientific research. Although current estimates suggest $\Lambda < 10^{-55} cm^{-2}$, nevertheless despite its tiny value Λ has important consequences on the large scale structure of the spacetime. Although it is impossible to summarize all the scientific work on Einsteins equations with a non vanishing Λ here, a great deal of effort has been focused on a family of a stationary- axisymmetric solutions of Einsteins equations with a non vanishing Λ, discovered long ago by Carter [3],[4]. These solutions besides Λ, contain two additional parameters (M, a). In the limit of vanishing Λ, the solutions reduce to the Kerr family of metrics while for $a = 0$, the Schwarzschild-(anti) de Sitter family is recovered. Due to these properties, M is interpreted as a mass[2] and a as a rotation parameter[3]. For certain values of the parameters (Λ, M, a) the solutions are interpreted

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[1] For a recent account of the history of the Λ-term the reader is referred to an enjoyable and well documented article by Straumann in [1], see also section (6.6) in [2].

[2] Whether this M can be rigorously interpreted as some form of mass energy runs into the subtleties in defining mass energy for asymptotically (anti) de Sitter spacetimes. For recent avances consult refs. [5], [6].

[3] Preliminary investigations show that (M, a) may admit a representation in terms of scalar polynomial curvature invariants patterning the same trend as for the case of Kerr (see the recent work in [7]). The scalar polynomial curvature invariants (Q_1, Q_2) (see [7],[8] for their precise form) locate the horizons and ergosurfaces for the Kerr-(anti) de Sitter metric. This interesting result was not realized at

as representing stationary axisymmetric black holes in asymptotically (anti)-de Sitter background[4] and in contrast to black holes in an asymptotically flat spacetime, these black holes may possess up to four horizons. Two of these horizons are cosmological and the other two are the inner and outer black hole horizons enclosing a ring-like singularity. It is believed that this Kerr-(anti) de Sitter family, may enjoy a uniqueness property as is the case of the Kerr black hole and thus interpreted as the final end state of the complete gravitational collapse of a bounded system in an asymptotically-(anti) de Sitter spacetime.
The behavior of geodesics on this space-times has been the subject of many investigations, see for example [13] [14] [15] [16] [17], while for the extension of the solution to arbitrary spacetime dimension see [18]. Effects of gravitational lensing on these spacetimes has been addressed in [19],[20],[21]. In two recent works [22],[23], the global structure of this family has been addressed. In [22], the authors introduced the notion of the projection diagrams as an alternative to Carter-Penrose conformal diagrams and through these diagrams, the structure of families of two dimensional submanifolds of the Kerr-(anti) de Sitter spacetime were investigated. In [23], the authors advanced an interesting interpretation of the Kerr-de Sitter spacetime, and they raised the question regarding the conditions upon (Λ, M, a) so that in a Kerr-de Sitter spacetime a coalescence between the inner and outer black hole hortizons takes place. Via a numerical example, they argue that for the case where $\Lambda > 0$ the condition $M^2 = a^2$ does not any longer characterize an extreme Kerr-de Sitter black hole.

A complete understanding of the global structure of the family of Kerr-(anti) de Sitter requires an investigation of the parameter space (Λ, M, a). Under what restrictions upon (Λ, M, a) does a Kerr-(anti) de Sitter spacetime describe a black hole with a well separated inner- outer and cosmological horizons? Does there exist a non trivial subset of the parameter space where a Kerr-(anti) de Sitter spacetime describes extreme configurations i.e. configurations where for instance the cosmological horizon coincides with the outer black horizon or do there exist super extreme configurations where three horizons coincide? If such configurations exist, do the parameters (Λ, M, a) retain values so that these configurations are important in astrophysics?
The purpose of the present work is to settle some of these questions. As a first step, we study the behavior of the roots of a quartic polynomial equation as function of the parameters (Λ, M, a). We treat this problem via the properties of the discriminate of polynomial equations and their relations to the determinant of the Sylvester matrix. Primarily, we focuse our attention to the parameter space which is of relevance for the description of astrophysical sources. Due to the currently suggested tiny value of Λ, the restrictions $M^2\Lambda \ll 1$ or (-and) $a^2\Lambda \ll 1$ cover many sources of astrophysical relevance[5].

The organization of this article is as follows. In the next section, we briefly introduce the family of Kerr-(anti) de Sitter metrics and identify the curvature and the coordinate singularities. In section III, we discus the roots of a polynomial equation $\Delta(r) = 0$ and relate their occurrence to the values of the parameters (Λ, M, a). In section IV, we comment on the global structure of the Kerr-de Sitter spacetimes and discusse future work and open problems.

II. THE KERR-DE SITTER METRIC

In a set of local $(t, \varphi, r, \vartheta)$ Boyer-Lindquist coordinates, the Kerr-(anti) de Sitter family of metrics has the form

$$g = -\frac{\Delta(r)}{I^2\rho^2}[dt - asin^2\theta d\varphi]^2 + \frac{\Delta(\vartheta)\sin^2\vartheta}{I^2\rho^2}[adt - (r^2+a^2)d\varphi]^2 + \frac{\rho^2}{\Delta(r)}d^2r + \frac{\rho^2}{\Delta(\vartheta)}d^2\vartheta \tag{1}$$

where

$$\rho^2 := r^2 + a^2\cos^2\vartheta, \quad \Delta(r) := -\frac{1}{3}\Lambda r^2(r^2+a^2) + r^2 - 2Mr + a^2, \quad \Delta(\vartheta) := 1 + \frac{1}{3}\Lambda a^2 cos^2\vartheta, \quad I := 1 + \frac{1}{3}\Lambda a^2.$$

For $\Lambda > 0$, this g is the Kerr-de Sitter metric while for $\Lambda < 0$ corresponds to the Kerr-anti de Sitter metric, (M, a^2) are free parameters while the factor I ensures the regularity of the g along the axis of axial symmetry. The fields $\xi_t = \frac{\partial}{\partial t}$ and $\xi_\varphi = \frac{\partial}{\partial\varphi}$ are commuting Killing vector fields with the zeros of the latter defining the rotation axis.

the time [7] was written.

[4] For arguments supporting this interpretation, see for instance ref.[9]. For progress on the important issue of defining a black hole on a cosmological spacetime, see for instance [10],[11],[12].

[5] Based on a value $\Lambda \sim 10^{-55}cm^{-2}$, then for the case of the Sun in a uniform rotation, $M^2\Lambda \sim 10^{-45}$ and $a^2\Lambda \sim 10^{-46}$ implying that the restrictions $M^2\Lambda \ll 1$ or (-and) $a^2\Lambda \ll 1$ leave plenty of room for the descriptions of astrophysical configurations.

Via algebraic manipulations using $GRTensorII$ [24], we find

$$C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} = \frac{48M^2}{\rho^{12}}F(r,\vartheta), \quad F(r,\vartheta) = (r^2 - a^2cos^2\vartheta)(\rho^4 - 16a^2r^2cos^2\vartheta), \tag{2}$$

$$C^*_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} = \frac{96M^2ra}{\rho^{12}}F^*(r,\vartheta), \quad F^*(r,\vartheta) = (r^2 - 3a^2cos^2\vartheta)(-3r^2 + a^2cos^2\vartheta)cos\vartheta, \tag{3}$$

where $C_{\mu\nu\lambda\rho}$ stand for the components of the Weyl, while $C^*_{\mu\nu\lambda\rho}$ denote the dual components. These invariants show that the curvature of (1) becomes unbounded as $\rho \to 0$ i.e. as the ring ($r = 0$, $\vartheta = \frac{\pi}{2}$) is approached. Remarkably, Λ drops out of these invariants and so they exhibit the same structure as the one exhibited by the Kerr metric. More remarkably the polynomial curvature invariants (Q_1, Q_2) (see [7]) have a very simple form and also locate the ergosurfaces and horizons.

Coordinate singularities in (1) occur along the axis of axial symmetry and these singularities can be removed by employing generalized Kerr-Schild coordinates. The other family of coordinate singularities[6] in (1), occur at the roots of $\Delta(r) = 0$ and further ahead we discuss the extension of (1) through these singularities.

Most of the analysis in the literature has focused on the Kerr- de Sitter metric subject to the assumption that the parameters (Λ, M, a) in (1), are chosen so that the quartic polynomial

$$\Delta(r) = -\frac{1}{3}\Lambda r^2(r^2 + a^2) + r^2 - 2Mr + a^2, \quad r \in \mathbb{R} \tag{4}$$

has one negative zero and the three distinct positive ones. Although for this case the Kerr-de Sitter family of spacetimes exhibits rich structure, nevertheless this family contains other configurations as well. A complete classification of all possible configurations, requires an understanding of the roots of the quartic $\Delta(r) = 0$ as a function of the parameters (Λ, M, a) and in the next section, we discuss that problem.

III. ON THE ROOTS OF THE EQUATION $\Delta(r) = 0$

It is convenient for the purposes of this section, to introduce a set of abbreviations so that

$$\Delta(r) = -\frac{1}{3}\Lambda r^4 + (1 - \frac{1}{3}\Lambda a^2)r^2 - 2Mr + a^2 := p_4r^4 + p_3r^3 + p_2r^2 + p_1r + p_0, \quad r \in \mathbb{R} \tag{5}$$

$$p_4 = -\frac{1}{3}\Lambda := L, \quad p_3 = 0, \quad p_2 = 1 - \frac{1}{3}\Lambda a^2 := N, \quad p_1 = -2M := K, \quad p_0 = a^2. \tag{6}$$

Furthermore, here after, $\Delta^{'}(r)$, $\Delta^{''}(r)$ stand for the first and second derivatives of the polynomial $\Delta(r)$ and $r_i, i \in (1, 2, 3, 4)$ denote the roots of $\Delta(r) = 0$.

The discriminant $D(\Delta_r)$ of the polynomial equation $\Delta(r) = 0$ is defined by

$$D(\Delta_r) = p_4^6 \prod_{i<j}(r_i - r_j)^2, \quad i, j \in (1, 2, 3, 4) \tag{7}$$

and satisfies the important relation:

$$D(\Delta_r) = \frac{R(\Delta, \Delta^{'})}{p_4} \tag{8}$$

where $R(\Delta, \Delta^{'})$ is the determinant of the Sylvester matrix associated with the polynomials $\Delta(r)$ and $\Delta^{'}(r)$. The determinant of the Sylvester matrix can be computed in terms of the coefficients of $\Delta(r)$ and $\Delta^{'}(r)$ and thus (7,

[6] For the case of Kerr-anti de Sitter ($\Lambda < 0$) an additional coordinate singularity may arise at the zeros of $\Delta(\vartheta)$ factor. The nature of this coordinate singularity, as well as more complete analysis of the Kerr-anti de Sitter spacetime, will be discussed elsewhere [25].

8) provide insights regarding the reality and multiplicity of the roots of $\Delta(r) = 0$. (For an introduction to the theory leading to the fundamental identity (8) see for instance [26],[27].) Although the evaluation of the Sylvester determinant can be a tedious job, fortunately for polynomials of low order, it has been tabulated and the result are readily available in the literature. For the polynomial $\Delta(r)$, consulting Maple, Mathematica, or [26],[27], we find

$$D(\Delta_r) = 256a^6L^3 - 128a^4N^2L^2 + 144a^2K^2NL^2 + 16a^2N^4L - 27K^4L^2 - 4K^2N^3L$$

$$= 128[2a^6L^3 - a^4N^2L^2 + \frac{9}{8}a^2K^2NL^2 + \frac{1}{8}a^2N^4L - \frac{27}{128}K^4L^2 - \frac{1}{32}K^2N^3L] \tag{9}$$

while the discriminants for $\Delta^{'}(r)$ and $\Delta^{''}(r)$ have the form

$$D(\Delta_r^{'}) = -16[27K^2L^2 + 8N^3L], \quad D(\Delta_r^{''}) = -96LN. \tag{10}$$

Hereafter we consider only the cases where $\Lambda > 0$ and $a \neq 0$. The analysis of Kerr-anti de Sitter is discussed elsewhere [25]. For $\Lambda > 0$ and $a \neq 0$, the equation $\Delta(r) = 0$ has at least one negative and one positive root. The widely discussed case of Kerr-de Sitter metric, assumes that the equation $\Delta(r) = 0$ admits one negative and three distinct positive roots and this occurs provided

$$D(\Delta_r) > 0, \quad \Lambda > 0, \quad a \neq 0. \tag{11}$$

The condition $D(\Delta_r) > 0$ by itself implies that either all the roots of $\Delta(r) = 0$ are real and distinct or they form two pairs of complex conjugate roots. However this last possibility is eliminated once the three conditions in (11) are taken together.

To get insights into the nature of the restrictions that conditions (11) impose upon (Λ, M, a^2), at first we write the discriminant $D(\Delta_r)$ in (9) in the equivalent form

$$D(\Delta_r) = -\frac{128}{3}[AM^4 + BM^2 + C], \quad A = \frac{9}{8}\Lambda^2, \quad B = -\frac{N\Lambda}{8}(N^2 + 12a^2\Lambda), \quad C = [\frac{2}{9}a^4\Lambda^2 + \frac{1}{3}a^2\Lambda N^2 + \frac{N^4}{8}]a^2\Lambda, \tag{12}$$

while a computation shows

$$B^2 - 4AC = \frac{\Lambda^2}{64}[N^6 - 12a^2\Lambda N^4 + 48a^4\Lambda^2N^2 - 64a^6\Lambda^3] \equiv \frac{\Lambda^2}{64}T(a^2\Lambda) \tag{13}$$

where $T(a^2\Lambda)$ is a sixth order polynomial with respect to the positive variable $a^2\Lambda$. The graph of this polynomial, determines domains on the $a^2\Lambda$-axis, where it is positive definite, negative definite or zero. For any $a^2\Lambda$ within the domains where $T(a^2\Lambda)$ is positive definite, $B^2 - 4AC$ is positive definite and thus $AM^4 + BM^2 + C = 0$ has real positive roots $\rho_-(\Lambda, a^2) < \rho_+(\Lambda, a^2)$. For any M^2 subject to the bounds

$$\rho_-(\Lambda, a^2) < M^2 < \rho_+(\Lambda, a^2) \tag{14}$$

the inequality $D(\Delta(r)) > 0$ holds. However since the observational data suggest a tiny value for Λ, it is reasonable to focus our attention to the case where a^2 and Λ are chosen so that $a^2\Lambda \ll 1$. Although this is a strong restriction, neverthesess it is satisfactory from the astrophysical view point since it covers wide range of astrophysical sources. Assuming therefore that $a^2\Lambda \ll 1$, the roots $\rho_-(\Lambda, a^2) < \rho_+(\Lambda, a^2)$ of $AM^4 + BM^2 + C = 0$ are

$$\rho_+(\Lambda, a^2) = \frac{1}{9\Lambda}[1 + 2a^2\Lambda + O(a^2\Lambda)^2)], \quad \rho_-(\Lambda, a^2) = a^2[1 + O(a^2\Lambda)^2] \tag{15}$$

implying that $D(\Delta(r)) > 0$, provided M^2 lies in the domain:

$$a^2[1 + O(a^2\Lambda)^2] < M^2 < \frac{1}{9\Lambda}[1 + 2a^2\Lambda + O(a^2\Lambda)^2)]. \tag{16}$$

This condition as far as we are aware, is new. It is fundamental and asserts that as long as M^2 is chosen to satisfy these bounds and $a^2\Lambda \ll 1$, then $\Delta(r) = 0$, admits three real positive distinct roots and a negative one. This estimate gives a relation between the mass M, rotation parameter a^2 and Λ so that a Kerr-de Sitter spacetime describes a black hole possessing an inner, outer and two cosmological horizons.

A modification of the conditions in (11) covers the case where $\Delta(r) = 0$ admits one negative, one positive and a pair of complex conjugate roots. This can occur, provided

$$D(\Delta_r) < 0, \quad \Lambda > 0, \quad a \neq 0. \tag{17}$$

Based on the same reasoning as above, we assume $a^2\Lambda \ll 1$ and thus the condition $D(\Delta_r) < 0$ holds, provided either $M^2 > \frac{1}{9\Lambda}$ or $M^2 < a^2$. A Kerr-de Sitter metric with parameters in that range describe a ring like curvature singularity enclosed between a pair of cosmological horizons. It is interesting to note that the parameter space is dominated by regions where the equation $\Delta(r) = 0$ has only a pair of real roots and this property has some interesting ramifications regarding the validity of cosmic censorship within the cosmological domain. However it is important to stress that the dominance of the parameter space by regions where $\Delta(r) = 0$ possess a pair of real roots holds under validity of the restriction $a^2\Lambda \ll 1$. Dropping this restriction likely will alter this conclusion. Although it is interesting to analyze the case where the condition $a^2\Lambda \ll 1$ is relaxed, we shall not proceed with this case any further here (see however, comments further ahead).

For completeness, we now investigate the case where $\Delta(r) = 0$ admits multiple roots and as a first case we treat the case where $\Delta(r) = 0$ admits a negative root and a positive root of multiplicity three (or the closely related alternative of a negative root of multiplicity three and a simple positive root). From the properties of the discriminant, it is easily seen that this setting occurs provided:

$$D(\Delta_r) = D({\Delta'}_r) = 0, \quad D({\Delta''}_r) > 0, \quad \Lambda > 0, \quad a \neq 0. \tag{18}$$

The conditions $D(\Delta_r) = D({\Delta'}_r) = 0$ guarantee that $\Delta(r) = 0$ has a real root r_i of multiplicity at least three, while $D({\Delta''}_r) > 0$ implies that r_i has multiplicity three.

Since $D(\Delta_r^{'}) = -16L[27K^2L + 8N^3] = \frac{64\Lambda}{3}[-9M^2\Lambda + 2N^3]$ and $\Lambda > 0$, clearly $D(\Delta_r^{'}) = 0$ cannot be satisfied unless $N > 0$. In turn, $N > 0$ requires $a^2\Lambda < 3$ and under validity of this constraint[7], $D(\Delta_r^{'}) = 0$ demands

$$M^2 = \frac{2N^3}{9\Lambda} = \frac{2}{9\Lambda}(1 - \frac{1}{3}a^2\Lambda)^3. \tag{19}$$

Since $\Delta''(r_i) = 0$ has $r_\pm = \pm(\frac{N}{2\Lambda})^{\frac{1}{2}}$ as its roots, there exist two possibilities regarding the triple root of $\Delta(r) = 0$. Choosing $r_i := r_+ = (\frac{N}{2\Lambda})^{\frac{1}{2}} > 0$ then $\Delta'(r_+) = 0$ provided the positive root is taken in (19) ie

$$M_+ = \frac{2^{\frac{1}{2}}}{3}\frac{N^{\frac{3}{2}}}{\Lambda^{\frac{1}{2}}}, \tag{20}$$

while $r_i := r_- = -(\frac{N}{2\Lambda})^{\frac{1}{2}} < 0$ obeys $\Delta'(r_-) = 0$ provided

$$M_- = -\frac{2^{\frac{1}{2}}}{3}\frac{N^{\frac{3}{2}}}{\Lambda^{\frac{1}{2}}}. \tag{21}$$

Finally r_+ (respectively r_-) is also a root of $\Delta(r) = 0$, provided

$$a^2 = \frac{N^2}{4\Lambda} = \frac{1}{4\Lambda}(1 - \frac{1}{3}a^2\Lambda)^2. \tag{22}$$

Setting $y = a^2\Lambda$, this constraint yields to the quadratic equation $y^2 - 42y + 9 = 0$ with roots $2y = 42 \pm \sqrt{1728} \simeq 42 \pm 41.56$ and thus for any choice of $\Lambda > 0$, exist a value for $a^2\Lambda$ consistent with the constraint $a^2\Lambda < 3$. Moreover for the values M^2 and a^2 as in (19, 22), it can be seen that the discriminant $D(\Delta_r)$ vanishes identically. In summary, the Kerr de Sitter family allows configurations where the inner, outer and cosmological horizon coincide. The location of this triple horizon and the required (positive mass) are[8] :

$$r_+ = (\frac{N}{2\Lambda})^{\frac{1}{2}}, \quad M_+ = \frac{2^{\frac{1}{2}}}{3}\frac{N^{\frac{3}{2}}}{\Lambda^{\frac{1}{2}}}. \tag{23}$$

Due to the tiny value of the observed Λ and since $N \simeq 1$, the occurrence of a triple root requires extremely high values of the mass parameters M^2 and likely these configurations are irrelevant for the description of astrophysical systems.

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[7] In the alternative case, i.e. whenever $a^2\Lambda \geq 3$ and $\Lambda > 0$, the discriminant $D(\Delta_r^{'})$ is always negative definite which implies further that $\Delta(r) = 0$ has only a pair of real roots.

[8] For any choice of $\Lambda > 0$, equation (22) determines a value of $a^2\Lambda$ and thus for this value of $a^2\Lambda$, (23) determine the location of the triple root and enclosed mass.

We finish this section by considering the case where $\Delta(r)=0$ has real roots but one of them has multiplicity two[9]. This arrangement can occur in one of the forms;

$$r_1=r_2<r_3<r_4,\quad r_1<r_2=r_3<r_4,\quad r_1<r_2<r_3=r_4. \tag{24}$$

Again, in view of the properties of the discriminant, this setting occurs provided

$$D(\Delta_r)=0,\quad D(\Delta'_r)>0,\quad \Delta(\hat{r}_i)=0,\quad \Delta^{''}(\hat{r}_i)\neq 0 \tag{25}$$

where $\hat{r}_i$ stands for any of the roots of $\Delta'(r)=0$ (assuming for the moment all of them real and distinct). The condition $D(\Delta_r)=0$ guarantees that $\Delta(r)=0$ admits (at least one) multiple root, $D(\Delta'_r)>0$ guarantees that $\Delta'(r)=0$ has three real and distinct roots , while $\Delta(\hat{r}_i)=0$ combined with $\Delta^{''}(\hat{r}_i)\neq 0$ implies that $\hat{r}_i$ is just a double root of $\Delta(r)=0$.

If $\Delta(r)=0$ and $\Delta'(r)=0$ share a common root denoted by $\hat{R}$, then necessarily (M,a^2) are related to this root via

$$M=\hat{R}(N-\frac{2\Lambda}{3}\hat{R}^2),\quad a^2=\hat{R}^2(N-\Lambda\hat{R}^2). \tag{26}$$

The first relation $M(\Lambda,\hat{R})$ is just a restatement that $\hat{R}$ is a root of $\Delta'(r)=0$ while $a^2(\Lambda,\hat{R})$, is the necessary and sufficient condition that $\hat{R}$ is a root of $\Delta(r)=0$ given that $\hat{R}$ is a root of $\Delta'(r)=0$. For these choices, $\Delta(r)$ has a double zero and thus $D(\Delta_r)=0$.
For $\Lambda>0$, the condition $D(\Delta'_r)>0$ requires

$$a^2\Lambda<3,\quad M^2<\frac{2}{9\Lambda}(1-\frac{1}{3}a^2\Lambda)=\frac{2}{9}\frac{N^3}{\Lambda} \tag{27}$$

and under these restrictions, $\Delta^{'}(r)=-\frac{4\Lambda}{3}[r^3+c_1r+c_0]:=-\frac{4\Lambda}{3}c(r)=0,\ c_1=-\frac{3N}{2\Lambda},\ c_0=\frac{3M}{2\lambda}$, possess three real roots[10]

$$\hat{r}_1=2\hat{\rho}^{\frac{1}{3}}cos(\frac{\hat{\vartheta}}{3}),\quad \hat{r}_2=2\hat{\rho}^{\frac{1}{3}}cos(\frac{\hat{\vartheta}}{3}+\frac{2\pi}{3}),\quad \hat{r}_3=2\hat{\rho}^{\frac{1}{3}}cos(\frac{\hat{\vartheta}}{3}+\frac{4\pi}{3}) \tag{28}$$

with $\hat{\rho}^2$ and the phase angle $\hat{\vartheta}$ given by:

$$\hat{\rho}^2=\frac{N^3}{8\Lambda^3}=\frac{1}{8\Lambda^3}(1-\frac{1}{3}\Lambda a^2)^3,\quad cos\hat{\vartheta}=-\frac{M}{x},\quad x^2=\frac{2}{9}\frac{N^3}{\Lambda}. \tag{29}$$

In order that any of the roots $\hat{r}_i$ in (28) is simultaneously a root of $\Delta(r)=0$, requires that the value of a^2 resulting from (26) once $\hat{R}$ is substituted for the chosen $\hat{r}_i$, to be positive definite and moreover be compatible with the constraints in (27). In order to get insights into the conditions leading to the appearance of double roots, we treat the case where $M^2\Lambda\ll 1$ and $a^2\Lambda\ll 1$. In that regime, the phase angle $\hat{\vartheta}$ in (29) can be approximated by

$$\hat{\vartheta}=\frac{\pi}{2}+\frac{M}{x}+O(\frac{M}{x})^2,\quad x^2=\frac{2}{9}\frac{N^3}{\Lambda} \tag{30}$$

and if we assume $M>0$, the roots in (28) can be approximated by:

$$\hat{r}_1=\sqrt{\frac{3}{2}}\frac{1}{\sqrt{\Lambda}}[1-\frac{M\sqrt{\Lambda}}{\sqrt{6}}+O(\frac{M}{x})^2],\quad \hat{r}_2=-\sqrt{\frac{3}{2}}\frac{1}{\sqrt{\Lambda}}[1+\frac{M\sqrt{\Lambda}}{\sqrt{6}}+O(\frac{M}{x})^2],\quad \hat{r}_3=M(1+O(\frac{M}{x})^2). \tag{31}$$

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[9] The possibility that there exist two roots both of multiplicity two it is not compatible with $\Lambda>0$ and $a^2\neq 0$.

[10] If r_0 is root of $c(r)=0$, then via Cardano's method we set $r_0=u+v$ and introduce (α,γ) so that $\alpha=u^3,\gamma=v^3$. In this representation r_0 is a root of $c(r)=0$, provided $\alpha\gamma=(-\frac{c_1}{3})^3$ and $\alpha+\gamma+c_0=0$ and thus (α,γ) are the roots of $x^2+c_0x+(-\frac{c_1}{3})^3=0$, $x\in\mathbb{R}$. If $\hat{\Delta}$ is the discriminant of this equation, then the requirement $D(\Delta'_r)>0$ implies $\hat{\Delta}<0$ and thus $2\alpha=-c_0+i\sqrt{|\hat{\Delta}|}$ while γ is the complex conjugate of α. In polar representation, $\alpha=\hat{\rho}e^{i\hat{\vartheta}}$ where $\hat{\rho}$ and $\hat{\vartheta}$ are as in (29) with the angle $\hat{\vartheta}$ measured counterclockwise from the positive real axis. The three real roots of $\Delta^{'}(r):=-\frac{4\Lambda}{3}c(r)=0$ are then the three distinct fractional powers: $\alpha^{\frac{1}{3}}+\gamma^{\frac{1}{3}}$ which in polar represenations are as in (28).

Thus two roots are positive and one is negative, an expected conclusion based on the structure of the equation $\Delta^{'}(r) = 0$ for positive M. Upon substituting $\hat{r}_1$ or $\hat{r}_2$ into the right hand side (26), we obtain a negative value for a^2 and thus $\hat{r}_1$ or $\hat{r}_2$ cannot be the location of the double root. However, the choice $\hat{r}_3$ gives $a^2 = M^2(1 + O(\Lambda M^2))$ which is compatible with the constraints[11] in (27). Thus in the regime $M^2\Lambda \ll 1$ and $a^2\Lambda \ll 1$, there is the possibility of the occurrence of a double root at the value $\hat{r}_3 \simeq M$ provided $a^2 = M^2(1 + O(\Lambda M^2))$.

In order to complete the picture regarding the formation of double roots, we examine the case where M approaches the limiting value x from bellow. Recalling that x^2 is defined in (30), and setting $M = x(1 - \epsilon)$ with $0 < \epsilon \ll 1$ then (28) in this regime yields the approximated roots

$$\hat{\vartheta} = \pi - \sqrt{2\epsilon}, \quad \epsilon > 0 \tag{32}$$

$$\hat{r}_1 = \frac{1}{\sqrt{2\Lambda}}[1 - \sqrt{\frac{2\epsilon}{3}} + O(\epsilon)], \quad \hat{r}_2 = \frac{2}{\sqrt{2\Lambda}}[-1 + O(\epsilon)], \quad \hat{r}_3 = \frac{1}{\sqrt{2\Lambda}}[1 + \sqrt{\frac{2\epsilon}{3}} + O(\epsilon)]. \tag{33}$$

However, $\hat{r}_2$ cannot be a double of $\Delta(r) = 0$, since the resulting a^2 turns out to be negative. For the other two roots we get $a^2 = (4\Lambda)^{-1}(1 + O(\epsilon))$ which suggests within our approximation, $(\hat{r}_1, \hat{r}_3)$ could be the location of a double root for $\Delta(r) = 0$. In summary therefore and for values of M^2 approaching the scale Λ^{-1} from bellow, there is the possibility of the occurrence of a double root in $\Delta(r) = 0$ at cosmological length scales.
We conclude this section by comparing the results derived so far with known results valid for the Reissner-Nordstrom-de Sitter family of spacetimes. This family, originally discovered by Kotller [28] but also appear as a special case of Carter's family of metrics derived in [3],[4]. It has been widely discussed in the literature and for properties and references, see for instance [29],[30],[31]. In a suitable set of spherical coordinates, the global structure of this family is determined by the function $\hat{\Delta}(r)$ defined by

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 = \frac{\hat{\Delta}(r)}{r^2}, \quad \hat{\Delta}(r) =: -\frac{\Lambda}{3}r^4 + r^2 - 2Mr + Q^2. \tag{34}$$

This $\hat{\Delta}(r)$ can be obtained from eq. (5) by taking $N(r) := 1$ and replacing a^2 by Q^2 with the latter interpreted as the electric charge in the solution. Therefore the results of this section are also applicable for the Reissner-Nordstrom-de Sitter family of spacetimes.

Assuming $\Lambda > 0$ and $Q^2 \neq 0$, it is seen from eqs. (19) and (22) that $\hat{\Delta}(r) = 0$ admits a triple positive root provided (Λ, M, Q^2) obey the conditions: $9M^2\Lambda = 2$ and $4\Lambda Q^2 = 1$. These conditions agree with those obtained in [29], [30] and in the terminology of [30], this case is referred as the " ultra extreme " Reissner-Nordstrom-de Sitter spacetime. Setting $N(r) = 1$ and replacing a^2 by Q^2 in (13), we find that in the limit $Q^2\Lambda \ll 1$, that the eq. $\hat{\Delta}(r) = 0$ has three distinct positive real roots provided

$$Q^2[1 + O(Q^2\Lambda)^2)] < M^2 < \frac{1}{9\Lambda}[1 + 3Q^2\Lambda + O(Q^2\Lambda)^2)] \tag{35}$$

which to the required order agrees with the results in [30]. For M^2 away from these domain, but still within the regime $Q^2\Lambda \ll 1$, a Reissner-Nordstrom-de Sitter spacetime admits only a cosmological horizon, referred in [30] as the generic naked singularity case. For particular values (Λ, M, Q^2), the equation $\hat{\Delta}(r) = 0$ admits two positive roots with the one having multiplicity two. These configurations describe extreme Reissner-Nordstrom-de Sitter spacetimes where either the inner and outer black hole coincide or the outer horizon coincides with the cosmological horizon. Under the conditions, $M^2\Lambda \ll 1$ and $Q^2\Lambda \ll 1$ our results show that the first possibility occurs under the condition $Q^2 = M^2(1 + O(M^2\Lambda))$, while the second possibility requires $M^2\Lambda \sim 1$.

After the completion of this work, we become aware of a thesis [32] written by one of the authors in [22], where a detailed analysis of the roots of the eq. $\Delta(r) = 0$ is presented. The starting point in [32], is the quartic polynomial

$$P(x) = -x^4 + 3x^2 - 2\beta x + \gamma, \quad r = \sigma x, \quad \beta = \frac{3M}{\Lambda\sigma^3}, \quad \gamma = \frac{3a^2}{\Lambda\sigma^4}, \quad \sigma = (\frac{N}{\Lambda})^{\frac{1}{2}}, \tag{36}$$

[11] The analysis for $M < 0$ case yields similar results except that now in (31) two of the roots are negative and one is positive. The interpretation of these results are of course identical to those in case of $M > 0$.

which is equivalent to the polynomial $\Delta(r)$ in (5). The formulas (9) and (10), imply that the discriminants of $P(x)$ and its derivative $P'(x)$ are

$$D(P_x) = -16.27[\beta^4 - (4\gamma+1)\beta^2 + \frac{16}{27}\gamma^3 + \frac{8}{3}\gamma^2 + 3\gamma], \quad D(P'_x) = -27.48[\beta^2 - 2] \tag{37}$$

and in [27], an analysis of the roots $\beta^2_-(\gamma), \beta^2_+(\gamma)$ of the equation $D(P_x) = 0$ has been made. It is shown that as long as $M > 0$, and $\gamma \in [0, \frac{3}{4}]$, then $D(P_x) > 0$ provided $\beta^2_-(\gamma) < \beta^2 < \beta^2_+(\gamma)$ and this condition is the analogue of our eq. (14). The condition that $D(P'_x) = 0$ requires $\beta^2 = 2$ which is identical to our eq. (19) which resulted upon imposing $D(\Delta'_r) = 0$. We have checked that the conclusions reached in [32], are in accord with the results obtained in this section (within the approximation employed in this work). The advantage of the approach in [32] lies in the simple form of the polynomial $P(x)$ that allowed an analytical treatment of the roots of the equation $D(P_x) = 0$ in terms of the parameters (β, γ). The latter are however, complicated expressions of the parameters (Λ, M, a^2). In contrast, in this work we strived to obtain conditions upon (Λ, M, a^2) so that a Kerr-de Sitter spacetime could be employed to model astrophysical sources, naturally therefore our analysis has been restricted to a limiter region of the parameter space.
Finally in [15], by a combination of analytical and numerical methods, conditions upon (Λ, M, a^2) have found so that a Kerr-de Sitter spacetime describes a black hole embedded within two cosmological horizons. Although qualitatively the results in [13] agree with those obtained here, due to different methods and approximations no further comparison can be made.

IV. DISCUSSION

In this work, we have re-examined the Kerr-de Sitter family of spacetimes and the results add complimentary insights on this structurally rich family of spacetimes. The conclusion that whenever $a^2\Lambda \ll 1$ and $a^2[1+O(a^2\Lambda)^2)] < M^2 < \frac{1}{9\Lambda}[1+2a^2\Lambda+O(a^2\Lambda)^2)]$, then a Kerr-de Sitter metric describes a black hole within pair of cosmological horizons, illustrates the role of a positive cosmological constant upon the black hole structure. When M^2 approaches a^2 from above, the inner and outer black hole horizon tend to coalesce, while at the othe extreme i.e. as M^2 approaches the limiting lenght scale Λ^{-1} from bellow, the outer horizon tends to coalesce with the cosmological horizon. These conclusions show that a non vanishing positive cosmological constant sets limit on the black hole size in accord with results obtained in [10],[11],[12].
Even though our results juxtapose the Kerr-de Sitter family of spacetime with the familiar Kerr family, in addition they offer further insights on the global structure of these spacetimes. Starting from a local Boyer-Lindquist $(t, \varphi, r, \vartheta)$ set of coordinates with $r_i < r < r_{i+1}$ where r_i, r_{i+1} are two consecutive zeros of $\Delta(r)$, then in a set of ingoing Finkelstein coordinates $(v, \overleftarrow{\varphi}, r, \vartheta)$ defined by

$$dv = dt + \frac{I(r^2+a^2)}{\Delta_r}dr, \quad d\overleftarrow{\varphi} = d\varphi + \frac{Ia}{\Delta_r}dr \tag{38}$$

the Kerr-de Sitter metric in (1) takes the form:

$$g = -\frac{\Delta_r - a^2\Delta_\vartheta sin^2\vartheta}{I^2\rho^2}d^2v + \frac{2}{I}dvdr - 2\frac{a}{I}sin^2\vartheta d\overleftarrow{\varphi}dr - 2\frac{asin^2\vartheta[(r^2+a^2)\Delta_\vartheta - \Delta_r]}{I^2\rho^2}dvd\overleftarrow{\varphi} +$$

$$+ \frac{\rho^2}{\Delta_\vartheta}d^2\vartheta + \frac{\Delta_\vartheta(r^2+a^2)^2 - \Delta_r a^2 sin^2\vartheta}{I^2\rho^2}sin^2\vartheta d^2\overleftarrow{\varphi}. \tag{39}$$

This g is regular over points where $\Delta(r) = 0$ and by allowing the coordinates (v, r) to run over the entire real line, an extension of the Kerr-de Sitter metric is obtained. In this $(v, \overleftarrow{\varphi}, r, \vartheta)$ coordinates, the translational ξ_t and rotational ξ_φ Killing fields take the form $\xi_t = \frac{\partial}{\partial u}, \xi_\varphi = \frac{\partial}{\partial \overleftarrow{\varphi}}$ and the equation $g(\xi_t, \xi_t) = 0$ shows the existence of non trivial ergospheres. Their properties depend upon the nature of the zeros of $\Delta(r)$ and their significant will be discussed elsewhere. Killing Horizons are generated by the Killing field $\hat{\xi}_i = \xi_t + \Omega_i\xi_\varphi$ where as in the case of Kerr, Ω_i are appropriate constants. These fields become null precisely over the $r = r_i$ hypersurface and depending upon the values of (Λ, M, a^2) a Kerr-de Sitter spacetime may contain up to four Killing horizons[12]. The maximal extension of a Kerr-de Sitter spacetime is obtained by introducing a set of a outgoing Finkelstein coordinates and joining together these

[12] Promoting these Killing horizons to event horizons is subtle. For some arguments in that direction see [9].

incomplete spacetimes in the same manner as for the case a Kerr spacetime. Two dimensional conformal diagrams describing the causal structure of the rotation axis, can be found for instance in ref [9], [23], [22]. In particular in [22] conformal diagrams for two dimensional sections of the Kert-de Sitter are analyzed. Finally and in view of the comparison between the functions $\Delta(r)$ and $\hat{\Delta}(r)$, the horizon structure between a Kerr-de Sitter and a Reissner-Nordstrom-de Sitter spacetime exhibit similarities. Of course the singularity structure in these spacetimes exhibits different features.

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[1] N. Straumann, "The History of the cosmological constant problem," gr-qc/0208027.

[2] D. Giulini and N. Straumann, "Einstein's impact on the physics of the twentieth century," gr-qc/0507103.

[3] B. Carter, Commun. Math. Phys. 17, 233. (1970).

[4] B. Carter, in Black Holes, (Gordon and Breach , New York. (1973))

[5] L. B. Szabados and P. Tod, "A positive Bondi–type mass in asymptotically de Sitter spacetimes," arXiv:1505.06637 [gr-qc].

[6] P. T. Chrusciel and G. Nagy, "The Mass of space - like hypersurfaces in asymptotically anti-de Sitter space-times," Adv. Theor. Math. Phys. **5**, 697 (2002) [gr-qc/0110014].

[7] M. Abdelqader and K. Lake, "Invariant characterization of the Kerr spacetime: Locating the horizon and measuring the mass and spin of rotating black holes using curvature invariants," Phys. Rev. D **91**, no. 8, 084017 (2015) [arXiv:1412.8757 [gr-qc]].

[8] D. N. Page and A. A. Shoom, "Local Invariants Vanishing on Stationary Horizons: A Diagnostic for Locating Black Holes," Phys. Rev. Lett. **114**, no. 14, 141102 (2015) [arXiv:1501.03510 [gr-qc]].

[9] G. W. Gibbons and S. W. Hawking, "Cosmological event horizons, thermodynamics, and particle creation," Phys. Rev. D15, 2738, (1977)

[10] T. Shiromizu, K. Nakao, H. Kodama and K. Maeda,, "Can large black holes collide in de Sitter space-time?An infationary scenario of an inhomogeneous universe," Phys. Rev. D **47**, no. 8, R3099 (1993)

[11] S. A. Hayward, T. Shiromizu, and K. Nakao. "A cosmological constant limits the size of black holes," Phys. Rev. D **49**, no. 10, 5080 (1994)

[12] M. E. G. Clement, M. Reiris and W. Simon, "The area-angular momentum inequality for black holes in cosmological spacetimes," Class. Quant. Grav. **32**, no. 14, 145006 (2015) [arXiv:1501.07243 [gr-qc]].

[13] Z. Stuchlik, G. Bao, E. Ostgaard and S. Hledik, Phys. Rev. D58, 084003, (1998)

[14] Z. Stuchlik, S. Hledik. Class. Quant. Grav. 17, 4541, (2000)

[15] Z. Stuchlik and P. Slany, "Equatorial circular orbits in the Kerr-de Sitter space-times," Phys. Rev. D **69**, 064001 (2004)

[16] P. C. Poudel and U. Khanal, "Effective potential and geodesic motion in Kerr-de Sitter space-time," arXiv:1309.1685 [gr-qc].

[17] T. Zannias, "Properties of causal geodesics on a Kerr-(anti) de Sitter spacetime" (In preparation.)

[18] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, "The General Kerr-de Sitter metrics in all dimensions," J. Geom. Phys. **53**, 49 (2005) [hep-th/0404008].

[19] G. V. Kraniotis, "Gravitational lensing and frame dragging of light in the Kerr-Newman and the Kerr-Newman-(anti) de Sitter black hole spacetimes," Gen. Rel. Grav. **46**, no. 11, 1818 (2014) [arXiv:1401.7118 [gr-qc]].

[20] G. V. Kraniotis, "Precise analytic treatment of Kerr and Kerr-(anti) de Sitter black holes as gravitational lenses," Class. Quant. Grav. **28**, 085021 (2011) [arXiv:1009.5189 [gr-qc]].

[21] G. V. Kraniotis, "Frame-dragging and bending of light in Kerr and Kerr-(anti) de Sitter spacetimes," Class. Quant. Grav. **22**, 4391 (2005) [gr-qc/0507056].

[22] P. T. Chrusciel, C. R. Olz and S. J. Szybka, "Space-time diagrammatics," Phys. Rev. D **86**, 124041 (2012) [arXiv:1211.1718 [gr-qc]].

[23] S. Akcay and R. A. Matzner, "Kerr-de Sitter Universe," Class. Quant. Grav. **28**, 085012 (2011) [arXiv:1011.0479 [gr-qc]].

[24] This package runs within Maple. The GRTensorII software and documentation is distributed freely from the address http://grtensor.org

[25] K. Lake and T. Zannias, "Remarks on the global structure of Kerr- anti de Sitter spacetimes "(In preparation)

[26] D. Eberly,"Low-Degree Polynomial Roots," http://www.geometrictools.com/

[27] V. V. Prasolov,"Polynomials," Springer-Verlag , (2004)

[28] F. Kottler, Ann. Phys. (Leipzig), 410, (1918)

[29] L. J. Romans, "Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory," Nucl. Phys. B383, 395, (1992)

[30] D. R. Brill and S. A. Hayward,"Global structure of a black hole cosmos and its extremes," Class. Quant. Grav. **11**, 359 (1994)

[31] K. Lake, "Reissner-Nordstrom-de Sitter metric, the third law, and cosmic censorship," Phys. Rev. D **19**, no. 2, 421 (1979)

[32] Christa. R. Olz, "The global structure of Kerr de Sitter metrics," Diplomarbeit, Univ. Wien, (2013)